

EXISTENCE RESULTS FOR ONE-DIMENSIONAL FRACTIONAL EQUATIONS

MAREK GALEWSKI AND GIOVANNI MOLICA BISCI

ABSTRACT. In this note a critical point result for differentiable functionals is exploited in order to prove that a suitable class of one-dimensional fractional problems admits at least one non-trivial solution under an asymptotical behaviour of the nonlinear datum at zero. A concrete example of an application is then presented.

1. INTRODUCTION

Critical point theory has been very useful in determining the existence of solutions for integer order differential equations with some boundary conditions; see for instance, in the vast literature on the subject, the classical books [16, 20, 33, 36] and references therein. But until now, there are few results for fractional boundary value problems (briefly BVP) which were established exploiting this approach, since it is often very difficult to establish a suitable space and variational functional for fractional problems. In the literature there are some approaches connected with investigations of fractional boundary value problems with critical point theory methods which depend on the type of fractional derivative used. Although fractional calculus shares some common features with classical differential calculus, there are some obvious differences, for example in the context of integration by parts, see for example [23].

In this paper, overcoming the above mentioned difficulty, a new variational approach is provided to investigate the existence of solutions to the following fractional BVP, namely (F_f) and given by:

$$\begin{aligned} \frac{d}{dt} \left({}_0D_t^{\alpha-1} ({}_0^c D_t^\alpha u(t)) - {}_tD_T^{\alpha-1} ({}_t^c D_T^\alpha u(t)) \right) + f(u(t)) &= 0, \text{ a.e. } t \in [0, T] \\ u(0) &= u(T) = 0, \end{aligned}$$

Key words and phrases. Existence results, Variational Methods, Fractional problems.

2010 Mathematics Subject Classification. 58E05, 26A33, 34A08, 34B15, 45J05.

Typeset by L^AT_EX.

where $\alpha \in (1/2, 1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1 - \alpha$ respectively, ${}_0^cD_t^\alpha$ and ${}_t^cD_T^\alpha$ are the left and right Caputo fractional derivatives of order α respectively, and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Fractional equations appear in concrete applications in many fields such as, among the others, optimization, finance, stratified materials, conservation laws, ultra-relativistic limits of quantum mechanics, minimal surfaces, materials science and water waves. This is one of the reason why, recently, non-local fractional problems are widely studied.

An interesting physical case is briefly discussed in [12] where the authors are interested on the existence and multiplicity of solutions for the following problem

$$\begin{aligned} \frac{d}{dt} \left(D_\beta(u(t)) \right) + \nabla F(t, u(t)) &= 0, \text{ a.e. } t \in [0, T] \\ u(0) &= u(T) = 0, \end{aligned}$$

where

$$D_\beta(u(t)) := \frac{1}{2}({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))),$$

$\beta \in [0, 1)$, and $F : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ (with $N \geq 1$) is a suitable given function. This model describes, as stating, that the mass flux of a particle is related to the negative gradient via a combination of the left and right fractional integrals.

Engineering applications of fractional concepts are connected with viscoelastic models, stochastic dynamics and with recently developed fractional-order thermoelasticity [32]. In these elds the main use of fractional operators has been concerned with the interpolation between the heat flux and its time-rate of change, that is related to the well-known second sound effect. In other recent studies [19] a fractional, non-local thermoelastic model has been proposed as a particular case of the non-local, integral, thermoelasticity introduced in [10]. We would like to mention also work [9] where the authors extend the non-local model of fractional heat conduction to the case of of a purely elastic material accounting for the thermoelastic coupling.

On the other hand, is whether or not the existence results got in the classical context can be extended to the non-local framework of the fractional Laplacian type operators. In this paper, motivated by a theoretical point of view, we are interested on the one-dimensional setting, previously considered in several papers (see, for instance, the manuscripts [3, 5] and references therein).

More concretely, in Theorem 3.1 we prove the existence of one solution to problem (F_f) requiring a simple algebraic inequality condition

namely (S_G) ; see Remarks 4.1 and 4.2. A parametric version of this result is successively discussed in Theorem 3.2 in which, for small values of the parameter and requiring an additional asymptotical behaviour of the potential at zero if $f(0) = 0$, the existence of one non-trivial solution is achieved; see Remark 4.3.

Moreover, we deduce the existence of solutions for small positive values of the parameter such that the corresponding solutions have smaller and smaller energies as the parameter goes to zero; see, for more details, Remark 4.4.

The proof of Theorem 3.1 (as well as of Theorem 3.2) is based on variational techniques. Precisely, in the sequel we will perform the variational principle of Ricceri obtained in [22]. Moreover, for several related topics and a careful analysis of the abstract framework we refer to the recent monograph [14].

A special case of our results reads as follows:

Theorem 1.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative continuous function and $\alpha \in (1/2, 1]$. Assume that*

$$\lim_{\xi \rightarrow 0^+} \frac{f(t)}{t} = +\infty. \quad (S_0)$$

Then, for every

$$\mu \in \Lambda := \left(0, \frac{\Gamma(\alpha)^2 |\cos(\pi\alpha)| (2\alpha - 1)}{T^{2\alpha}} \left(\sup_{\gamma > 0} \frac{\gamma^2}{\int_0^\gamma f(s) ds} \right) \right),$$

the following parametric problem

$$\begin{aligned} \frac{d}{dt} \left({}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u(t)) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u(t)) \right) + \mu f(u(t)) &= 0, \text{ a.e. } t \in [0, T] \\ u(0) &= u(T) = 0, \end{aligned}$$

admits at least one non-trivial solution in E_0^α . Moreover, one has

$$\lim_{\mu \rightarrow 0^+} \int_0^T |{}_0^c D_t^\alpha u_\mu(t)|^2 dt = 0,$$

and the function

$$\mu \mapsto - \int_0^T {}_0^c D_t^\alpha u_\mu(t) \cdot {}_t^c D_T^\alpha u_\mu(t) dt - \mu \int_0^T \left(\int_0^{u_\mu(t)} f(s) ds \right) dt,$$

is negative and strictly decreasing in Λ .

We would like to emphasize that, as observed in Remark 3.1, the energy functional J_μ associated to the above parametric problem can be unbounded from below in the ambient space E_0^α . Hence, in order to find critical points of J_μ we can not argue, in general, by direct minimization techniques; see Example 4.1 and Remark 4.6.

Our assumptions also do not allow to use classical minimization topics and related arguments.

Following [12], we point out that there are few results on the solutions to fractional BVP which were established by the critical point theory, since it is often very difficult to establish a suitable space and variational functional for fractional differential equations with some boundary conditions. These difficulties are mainly caused by the following significative facts:

- (i) the composition rule in general fails to be satisfied by fractional integral and fractional derivative operators;
- (ii) the fractional integral is a singular integral operator and fractional derivative operator is non-local;
- (iii) the adjoint of a fractional differential operator is not the negative of itself.

It should be mentioned here that the fractional variational principles were started to be investigated deeply. The fractional calculus of variations was introduced by Riewe in [21] where he presented a new approach to mechanics that allows one to obtain the equations for a nonconservative system using certain functionals.

For completeness, we recall that a careful and interesting analysis of the elliptic fractional case was developed in the recent and nice works [24, 25, 26, 28] and references therein.

The paper is organized as follows. In Section 2 we give the principal definitions related to our abstract functional framework. In Section 3 we prove Theorems 3.1 and 3.2, while Section 4 is devoted to some comments on the results of the paper. Finally, an application of Theorem 3.2, is presented in Example 4.2 studying a one-dimensional fractional equation involving a suitable non-linearity.

2. SOME PRELIMINARIES

This section is devoted to the notations used along the paper. We also give some preliminary results which will be useful in the sequel.

2.1. The functional setting.

Definition 2.1. *Let u be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ for a function u*

are defined by

$${}_a D_t^{-\alpha} u(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds,$$

and

$${}_t D_b^{-\alpha} u(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) ds,$$

for every $t \in [a, b]$, provided the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma(\alpha)$ is the standard gamma function given by

$$\Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1} e^{-z} dz.$$

Set $AC^n([a, b], \mathbb{R})$ the space of functions $u : [a, b] \rightarrow \mathbb{R}$ such that $u \in C^{n-1}([a, b], \mathbb{R})$ and $u^{(n-1)} \in AC([a, b], \mathbb{R})$. Here, as usual, $C^{n-1}([a, b], \mathbb{R})$ denotes the set of mappings having $(n-1)$ times continuously differentiable on $[a, b]$. In particular we denote $AC([a, b], \mathbb{R}) := AC^1([a, b], \mathbb{R})$.

Definition 2.2. Let $\gamma \geq 0$ and $n \in \mathbb{N}$.

(i) If $\gamma \in (n-1, n)$ and $u \in AC^n([a, b], \mathbb{R})$, then the left and right Caputo fractional derivatives of order γ for function u denoted by ${}_a^c D_t^\gamma u(t)$ and ${}_t^c D_b^\gamma u(t)$, respectively, exist almost everywhere on $[a, b]$, ${}_a^c D_t^\gamma u(t)$ and ${}_t^c D_b^\gamma u(t)$ are represented by

$${}_a^c D_t^\gamma u(t) = \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-s)^{n-\gamma-1} u^{(n)}(s) ds,$$

and

$${}_t^c D_b^\gamma u(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \int_t^b (s-t)^{n-\gamma-1} u^{(n)}(s) ds,$$

for every $t \in [a, b]$, respectively.

(ii) If $\gamma = n-1$ and $u \in AC^{n-1}([a, b], \mathbb{R})$, then ${}_a^c D_t^{n-1} u(t)$ and ${}_t^c D_b^{n-1} u(t)$ are represented by

$${}_a^c D_t^{n-1} u(t) = u^{(n-1)}(t), \quad \text{and} \quad {}_t^c D_b^{n-1} u(t) = (-1)^{(n-1)} u^{(n-1)}(t),$$

for every $t \in [a, b]$.

With these definitions, we have the rule for fractional integration by parts, and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator, which were proved in [13] and [23].

Proposition 2.1. We have the following property of fractional integration

$$(1) \quad \int_a^b [{}_a D_t^{-\gamma} u(t)] v(t) dt = \int_a^b [{}_t D_b^{-\gamma} v(t)] u(t) dt, \quad \gamma > 0,$$

provided that $u \in L^p([a, b], \mathbb{R})$, $v \in L^q([a, b], \mathbb{R})$ and $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1 + \gamma$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \gamma$.

Proposition 2.2. *Let $n \in \mathbb{N}$ and $n - 1 < \gamma \leq n$. If $u \in AC^n([a, b], \mathbb{R})$ or $u \in C^n([a, b], \mathbb{R})$, then*

$${}_a D_t^{-\gamma}({}_a^c D_t^\gamma u(t)) = u(t) - \sum_{j=0}^{n-1} \frac{u^{(j)}(a)}{j!} (t-a)^j,$$

and

$${}_t D_b^{-\gamma}({}_t^c D_b^\gamma u(t)) = u(t) - \sum_{j=0}^{n-1} \frac{(-1)^j u^{(j)}(b)}{j!} (b-t)^j,$$

for every $t \in [a, b]$. In particular, if $0 < \gamma \leq 1$ and $u \in AC([a, b], \mathbb{R})$ or $u \in C^1([a, b], \mathbb{R})$, then

$${}_a D_t^{-\gamma}({}_a^c D_t^\gamma u(t)) = u(t) - u(a),$$

and

$${}_t D_b^{-\gamma}({}_t^c D_b^\gamma u(t)) = u(t) - u(b).$$

Remark 2.1. *We recall that a function $u \in AC([0, T], \mathbb{R})$ is said to be a solution of (F_f) if the map*

$$t \mapsto {}_0 D_t^{\alpha-1}({}_0^c D_t^\alpha u(t)) - {}_t D_T^{\alpha-1}({}_t^c D_T^\alpha u(t)),$$

is derivable (in the classical sense) for almost every $t \in [0, T]$, and

$$\begin{aligned} \frac{d}{dt} \left({}_0 D_t^{\alpha-1}({}_0^c D_t^\alpha u(t)) - {}_t D_T^{\alpha-1}({}_t^c D_T^\alpha u(t)) \right) + f(u(t)) &= 0, \text{ a.e. } t \in [0, T] \\ u(0) &= u(T) = 0. \end{aligned}$$

To establish a variational structure for (F_f) , it is necessary to construct appropriate function spaces.

Following [12], denote by $C_0^\infty([0, T], \mathbb{R})$ the set of all functions $g \in C^\infty([0, T], \mathbb{R})$ with $g(0) = g(T) = 0$.

Definition 2.3. *Let $0 < \alpha \leq 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty([0, T], \mathbb{R})$ with respect to the norm*

$$\|u\| := \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt \right)^{1/2}.$$

As observed in [12], the space E_0^α is a Hilbert space with norm

$$\|u\|_\alpha := \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt \right)^{1/2}.$$

For every $u \in E_0^\alpha$, set

$$\|u\|_{L^s} := \left(\int_0^T |u(t)|^s dt \right)^{1/s}, \quad (s \geq 1)$$

and

$$\|u\|_\infty := \max_{t \in [0, T]} |u(t)|.$$

The next result will be crucial in the sequel.

Lemma 2.1. *Assume $\alpha \in (1/2, 1]$ and let $\{u_j\} \subset E_0^\alpha$ be a sequence weakly convergent to $u \in E_0^\alpha$. Then, $u_j \rightarrow u$ in $C^0([0, T], \mathbb{R})$, i.e. $\|u_j - u\|_\infty \rightarrow 0$, as $j \rightarrow \infty$.*

See [12, Proposition 3.3].

Finally, one has the following two Lemmas.

Lemma 2.2. *Let $\alpha \in (1/2, 1]$. For every $u \in E_0^\alpha$, we have*

$$\begin{aligned} \|u\|_{L^2} &\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|{}_0^c D_t^\alpha u\|_{L^2}, \\ \|u\|_\infty &\leq \frac{T^{\alpha - \frac{1}{2}}}{\Gamma(\alpha) \sqrt{2\alpha - 1}} \|{}_0^c D_t^\alpha u\|_{L^2}. \end{aligned}$$

Lemma 2.3. *Let $\alpha \in (1/2, 1]$, then for every $u \in E_0^\alpha$, we have*

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \leq - \int_0^T {}_0^c D_t^\alpha u(t) \cdot {}_t^c D_T^\alpha u(t) dt \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2.$$

See [12] for details.

2.2. A critical points result for differentiable functionals. In order to prove our main result, stated in Theorem 3.1, in the following we will perform the variational principle of Ricceri established in [22]. For the sake of clarity, we recall it here below in the form given in [8].

Theorem 2.1. *Let Y be a reflexive real Banach space, and $\Phi, \Psi : Y \rightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive in Y and Ψ is sequentially weakly upper semicontinuous in Y . Let J_μ be the functional defined as $J_\mu := \Phi - \mu\Psi$, $\mu \in \mathbb{R}$, and for any $r > \inf_Y \Phi$ let φ be the function defined as*

$$\varphi(r) := \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for any $r > \inf_Y \Phi$ and any $\mu \in (0, 1/\varphi(r))^1$, the restriction of the functional J_μ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (precisely a local minimum) of J_μ in Y .

3. THE MAIN RESULTS

This section is devoted to the proof of the main result of the present paper, that is the following.

Theorem 3.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and $\alpha \in (1/2, 1]$. Set*

$$\kappa_\alpha := \frac{T^{2\alpha}}{\Gamma(\alpha)^2 |\cos(\pi\alpha)|(2\alpha - 1)},$$

where Γ is the Euler function. Assume that

$$\sup_{\gamma > 0} \frac{\gamma^2}{\max_{|\xi| \leq \gamma} \int_0^\xi f(t) dt} > \kappa_\alpha. \quad (S_G)$$

Then problem (F_f) admits at least one solution in E_0^α .

Proof. The idea of the proof consists in applying [8, Theorem 2.1; part a)] taking $Y := E_0^\alpha$.

Hence, for given $u \in E_0^\alpha$, we define functionals $\Phi, \Psi : E_0^\alpha \rightarrow \mathbb{R}$ as follows:

$$\Phi(u) := - \int_0^T {}^c_0 D_t^\alpha u(t) \cdot {}^c_t D_T^\alpha u(t) dt, \quad \text{and} \quad \Psi(u) := \int_0^T F(u(t)) dt,$$

where $F(\xi) := \int_0^\xi f(s) ds$, for every $\xi \in \mathbb{R}$.

Clearly, Φ and Ψ are continuously Gâteaux differentiable functional whose Gâteaux derivatives at the point $u \in E_0^\alpha$ are given by

$$\begin{aligned} \Phi'(u)(v) &= - \int_0^T ({}^c_0 D_t^\alpha u(t) \cdot {}^c_t D_T^\alpha v(t) + {}^c_t D_T^\alpha u(t) \cdot {}^c_0 D_t^\alpha v(t)) dt, \\ \Psi'(u)(v) &= \int_0^T f(u(t)) v(t) dt = - \int_0^T \int_0^t f(u(s)) ds \cdot v'(t) dt, \end{aligned}$$

for every $v \in E_0^\alpha$.

Moreover, it is easy to see that

$$\Phi'(u)(v) = \int_0^T ({}_0 D_t^{\alpha-1} ({}^c_0 D_t^\alpha u(t)) - {}_t D_T^{\alpha-1} ({}^c_t D_T^\alpha u(t))) \cdot v'(t) dt.$$

¹Note that, by definition, $\varphi(r) \geq 0$ for any $r > \inf_Y \Phi$. Here and in the following, if $\varphi(r) = 0$, by $1/\varphi(r)$ we mean $+\infty$, i.e. we set $1/\varphi(r) = +\infty$.

Thus, the functional $J := \Phi - \Psi \in C^1(E_0^\alpha, \mathbb{R})$ and the functionals Φ and Ψ are respectively sequentially weakly lower and upper semicontinuous. As concerns functional Φ , this follows by Lemma 2.1. Indeed Φ is strongly continuous and convex and hence sequentially weakly lower semicontinuous. Moreover, the weakly upper semicontinuity the functional Ψ can be proved arguing in a standard way by using again the compact embedding $E_0^\alpha \hookrightarrow C^0([0, T], \mathbb{R})$.

Further, from Lemma 2.3, the functional Φ is coercive. Indeed, one has

$$\Phi(u) := - \int_0^T {}^c D_t^\alpha u(t) \cdot {}^c D_T^\alpha u(t) dt \geq |\cos(\pi\alpha)| \|u\|_\alpha^2 \rightarrow +\infty,$$

as $\|u\|_\alpha \rightarrow +\infty$.

Moreover, a critical point of the functional J is a solution of (F_f) . Indeed, if $u_* \in E_0^\alpha$ is a critical point of J , then

$$(2) \quad \begin{aligned} 0 = J'(u_*)(v) &= \int_0^T \left({}_0 D_t^{\alpha-1} ({}^c D_t^\alpha u_*(t)) - {}_t D_T^{\alpha-1} ({}^c D_T^\alpha u_*(t)) \right. \\ &\quad \left. + \int_0^t f(u_*(s)) ds \right) v'(t) dt, \end{aligned}$$

for every $v \in E_0^\alpha$.

Now, we can choose $v \in E_0^\alpha$ such that

$$v(t) := \sin \frac{2k\pi t}{T} \quad \text{or} \quad v(t) := 1 - \cos \frac{2k\pi t}{T}, \quad (k = 1, 2, \dots).$$

The theory of Fourier series and (2) imply

$$(3) \quad {}_0 D_t^{\alpha-1} ({}^c D_t^\alpha u_*(t)) - {}_t D_T^{\alpha-1} ({}^c D_T^\alpha u_*(t)) + \int_0^t f(u_*(s)) ds = \kappa$$

a.e. on $[0, T]$ for some $\kappa \in \mathbb{R}$.

By (3), it is easy to show that $u_* \in E_0^\alpha$ is a solution of (F_f) . Now, we look on the existence of a critical point of the functional J in E_0^α .

Since condition (S_G) holds, there exists $\bar{\gamma} > 0$ such that

$$(4) \quad \frac{\bar{\gamma}^2}{\max_{|\xi| \leq \bar{\gamma}} \int_0^\xi f(t) dt} > \frac{T^{2\alpha}}{\Gamma(\alpha)^2 |\cos(\pi\alpha)| (2\alpha - 1)}.$$

Now, by Lemma 2.2 (note that $\alpha > 1/2$), for each $u \in E_0^\alpha$ we have

$$(5) \quad \|u\|_\infty \leq c \left(\int_0^T |{}^c D_t^\alpha u(t)|^2 dt \right)^{1/2} = c \|u\|_\alpha,$$

where

$$c := \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}.$$

Hence, set

$$(6) \quad r := \frac{|\cos(\pi\alpha)|}{c^2} \bar{\gamma}^2.$$

Moreover, for every $u \in E_0^\alpha$ such that $u \in \Phi^{-1}((-\infty, r))$, by Lemma 2.3 we have

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \leq \Phi(u) < r,$$

which implies

$$(7) \quad \|u\|_\alpha^2 < \frac{r}{|\cos(\pi\alpha)|}.$$

Thus, by (5), (6) and (7) we obtain

$$|u(t)| \leq c \|u\|_\alpha < c \sqrt{\frac{r}{|\cos(\pi\alpha)|}} = \bar{\gamma}, \quad \forall t \in [0, T].$$

Hence,

$$\Psi(u) = \int_0^T F(u(t)) dt \leq \int_0^T \max_{|\xi| \leq \bar{\gamma}} F(\xi) dt = T \max_{|\xi| \leq \bar{\gamma}} F(\xi),$$

for every $u \in E_0^\alpha$ such that $u \in \Phi^{-1}((-\infty, r))$.

Then,

$$\sup_{u \in \Phi^{-1}((-\infty, r))} \Psi(u) \leq T \max_{|\xi| \leq \bar{\gamma}} F(\xi).$$

Taking into account the above computations and remarks, one has the following inequalities

$$\begin{aligned} \varphi(r) &= \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v) - \Psi(u)}{r - \Phi(u)} \\ &\leq \frac{\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v)}{r} \\ &\leq \frac{c^2 T \max_{|\xi| \leq \bar{\gamma}} F(\xi)}{|\cos(\pi\alpha)| \bar{\gamma}^2}. \end{aligned}$$

Thus, it follows that

$$(8) \quad \varphi(r) \leq \kappa_\alpha \frac{\max_{|\xi| \leq \bar{\gamma}} F(\xi)}{\bar{\gamma}^2},$$

observing that

$$\kappa_\alpha = \frac{c^2 T}{|\cos(\pi\alpha)|}.$$

Consequently, by (4) and (8) one has $\varphi(r) < 1$. Hence, since $1 \in (0, 1/\varphi(r))$, Theorem 2.1 ensures that the functional J admits at least one critical point (local minima) $\tilde{u} \in \Phi^{-1}((-\infty, r))$. The proof is complete. \square

We explicitly note that Theorem 2.1 can be exploited proving the existence of one solution for the parametric version of problem (F_f) , namely (F_f^μ) and given by:

$$\begin{aligned} \frac{d}{dt} \left({}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u(t)) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u(t)) \right) + \mu f(u(t)) &= 0, \text{ a.e. } t \in [0, T] \\ u(0) &= u(T) = 0. \end{aligned}$$

More precisely, one has the following existence property.

Theorem 3.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and $\alpha \in (1/2, 1]$. Then, for every μ sufficiently small, i.e.*

$$\mu \in \left(0, \frac{1}{\kappa_\alpha} \left(\sup_{\gamma > 0} \frac{\gamma^2}{\max_{|\xi| \leq \gamma} F(\xi)} \right) \right),$$

problem (F_f^μ) admits at least one solution $u_\mu \in E_0^\alpha$.

Proof. Let us take

$$0 < \mu < \frac{1}{\kappa_\alpha} \left(\sup_{\gamma > 0} \frac{\gamma^2}{\max_{|\xi| \leq \gamma} F(\xi)} \right).$$

Hence, there exists $\bar{\gamma} > 0$ such that

$$(9) \quad \kappa_\alpha \mu < \frac{\bar{\gamma}^2}{\max_{|\xi| \leq \bar{\gamma}} F(\xi)}.$$

Set

$$(10) \quad r := \frac{|\cos(\pi\alpha)|}{c^2} \bar{\gamma}^2.$$

Preserving the notations as in the proof of Theorem 3.1, one has

$$\varphi(r) \leq \frac{\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v)}{r} \leq \frac{c^2 T}{|\cos(\pi\alpha)|} \frac{\max_{|\xi| \leq \bar{\gamma}} F(\xi)}{\bar{\gamma}^2} < \frac{1}{\mu}.$$

Hence, since $\mu \in (0, 1/\varphi(r))$, Theorem 2.1 ensures that the functional J_μ admits at least one critical point (local minima) $u_\mu \in \Phi^{-1}((-\infty, r))$. The proof is complete. \square

Remark 3.1. *Our variational approach consist in looking for critical points of the functional J_μ naturally associated with problem (F_f^μ) . We would like to note that, in general, J_μ can be unbounded from below in E_0^α .*

Indeed, for instance, in the case when $f(t) := 1 + |t|^{q-2}t$ with $q \in (2, +\infty)$, for any fixed $u \in E_0^\alpha \setminus \{0\}$ and $\tau \in \mathbb{R}$, we get

$$\begin{aligned} J_\mu(\tau u) &= \Phi(\tau u) - \mu \int_0^T F(\tau u(t)) dt \\ &\leq \frac{\tau^2}{|\cos(\pi\alpha)|} \|u\|_\alpha^2 - \mu\tau \|u\|_{L^1} - \frac{\mu\tau^q}{q} \|u\|_{L^q}^q \rightarrow -\infty \end{aligned}$$

as $\tau \rightarrow +\infty$.

Hence, in order to find critical points of J_μ we can not argue, in general, by direct minimization.

4. SOME COMMENTS

In this section we give some remarks and a concrete example of application of our results.

Remark 4.1. *If in Theorem 3.1 the function f is non-negative, condition (S_G) assumes the more simple and significative form*

$$\sup_{\gamma>0} \frac{\gamma^2}{\int_0^\gamma f(s)ds} > \kappa_\alpha. \quad (S'_G)$$

Moreover, if the following assumption is verified

$$\limsup_{\xi \rightarrow +\infty} \frac{\xi^2}{\int_0^\xi f(s)ds} > \kappa_\alpha, \quad (S_\infty)$$

then, condition (S'_G) automatically holds.

Remark 4.2. *Let $\bar{\gamma} > 0$ be a real constant such that*

$$\frac{\bar{\gamma}^2}{\max_{|\xi| \leq \bar{\gamma}} \int_0^\xi f(s)ds} > \kappa_\alpha,$$

and said $\tilde{u} \in E_0^\alpha$ be the solution of problem (F_f) obtained by using Theorem 2.1. Hence, since $\tilde{u} \in \Phi^{-1}((-\infty, r))$, it follows that $\|\tilde{u}\|_\infty \leq \bar{\gamma}$.

Remark 4.3. *If in Theorem 3.2 one has $f(0) \neq 0$, then the obtained solution is clearly non-trivial. On the other hand, the non-triviality of the solution can be achieved also in the case $f(0) = 0$ requiring the additional condition at zero*

$$(11) \quad \lim_{t \rightarrow 0^+} \frac{F(t)}{t^2} = +\infty.$$

Indeed, let $0 < \bar{\mu} < \mu^$. Then, there exists $\bar{\gamma} > 0$ such that*

$$(12) \quad \kappa_\alpha \bar{\mu} < \frac{\bar{\gamma}^2}{\max_{|\xi| \leq \bar{\gamma}} F(\xi)}.$$

Thanks to Theorem 2.1, for every $\mu \in (0, \bar{\mu})$ there exists a critical point of J_μ such that

$$u_\mu \in \Phi^{-1}((-\infty, r_{\bar{\mu}})),$$

where

$$r_{\bar{\mu}} := \frac{|\cos(\pi\alpha)|}{c^2} \bar{\gamma}^2,$$

In particular, u_μ is a global minimum of the restriction of J_μ to $\Phi^{-1}((-\infty, r_{\bar{\mu}}))$.

We will prove that the function u_μ cannot be trivial. To this end, let us show that

$$(13) \quad \limsup_{\|u\|_\alpha \rightarrow 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty.$$

Due to our assumptions at zero, we can fix a sequence $\{\xi_j\} \subset \mathbb{R}^+$ converging to zero and two constant σ , and κ (with $\sigma > 0$) such that

$$\lim_{j \rightarrow \infty} \frac{F(\xi_j)}{\xi_j^2} = +\infty,$$

and

$$F(\xi) \geq \kappa \xi^2,$$

for every $\xi \in [0, \sigma]$.

Now, fix a function $v \in E_0^\alpha$ (note that $C^\infty[0, T] \subset E_0^\alpha$) such that:

- i) $v(t) \in [0, 1]$, for every $t \in [0, T]$;
- ii) $v(t) = 1$, for every $t \in [\frac{T}{4}, \frac{3T}{4}]$.

Hence, fix $M > 0$ and consider a real positive number η with

$$M < \frac{\eta T + \kappa \int_{[0, T] \setminus [\frac{T}{4}, \frac{3T}{4}]} v(t)^2 dt}{\Phi(v)}.$$

Then, there is $\nu \in \mathbb{N}$ such that $\xi_j < \sigma$ and

$$\int_0^{\xi_j} f(s)ds \geq 2\eta\xi_j^2,$$

for every $j > \nu$.

At this point, for every $j > \nu$, and bearing in mind the properties of the function v ($0 \leq \xi_j v(t) < \sigma$ for j sufficiently large), one has

$$\begin{aligned} \frac{\Psi(\xi_j v)}{\Phi(\xi_j v)} &= \frac{\int_{[\frac{T}{4}, \frac{3T}{4}]} \left(\int_0^{\xi_j} f(s)ds \right) dt + \int_{[0, T] \setminus [\frac{T}{4}, \frac{3T}{4}]} F(\xi_j v(t)) dt}{\Phi(\xi_j v)} \\ &\geq \frac{\eta T + \kappa \int_{[0, T] \setminus [\frac{T}{4}, \frac{3T}{4}]} v(t)^2 dt}{\Phi(v)} > M. \end{aligned}$$

Since M could be arbitrarily large, it follows that

$$\lim_{j \rightarrow \infty} \frac{\Psi(\xi_j v)}{\Phi(\xi_j v)} = +\infty,$$

from which (13) clearly follows.

Hence, there exists a sequence $\{w_j\} \subset E_0^\alpha$ strongly converging to zero, such that, for every j sufficiently large, $w_j \in \Phi^{-1}((-\infty, r_\mu))$, and

$$J_\mu(w_j) := \Phi(w_j) - \mu\Psi(w_j) < 0.$$

Since u_μ is a global minimum of the restriction of J_μ to $\Phi^{-1}((-\infty, r_\mu))$, we conclude that

$$(14) \quad J_\mu(u_\mu) < 0,$$

so that u_μ is not trivial.

Remark 4.4. Put

$$\mu^* := \frac{1}{\kappa_\alpha} \left(\sup_{\gamma > 0} \frac{\gamma^2}{\max_{|\xi| \leq \gamma} F(\xi)} \right).$$

From (14) we easily see that the map

$$(15) \quad (0, \mu^*) \ni \mu \mapsto J_\mu(u_\mu) \text{ is negative.}$$

Further, we claim that

$$\lim_{\mu \rightarrow 0^+} \|u_\mu\|_\alpha = 0.$$

Indeed, let $0 < \bar{\mu} < \mu^*$. Thus, there exists $\bar{\gamma} > 0$ such that

$$(16) \quad \kappa_\alpha \bar{\mu} < \frac{\bar{\gamma}^2}{\max_{|\xi| \leq \bar{\gamma}} F(\xi)}.$$

Set

$$r_{\bar{\mu}} := \frac{|\cos(\pi\alpha)|}{c^2} \bar{\gamma}^2.$$

Bearing in mind that Φ is coercive and that for every $\mu \in (0, \bar{\mu}]$ the solution $u_\mu \in \Phi^{-1}((-\infty, r_{\bar{\mu}}))$, one has that there exists a positive constant L such that

$$\|u_\mu\|_\alpha \leq L,$$

for every $\mu \in (0, \bar{\mu}]$.

Then, one has that

$$(17) \quad \left| \int_0^T f(u_\mu(t)) u_\mu(t) dt \right| \leq cL \max_{|s| \leq cL} |f(s)| T,$$

for every $\mu \in (0, \bar{\mu}]$.

Since u_μ is a critical point of J_μ , then $J'_\mu(u_\mu)(v) = 0$, for any $v \in E_0^\alpha$ and every $\mu \in (0, \bar{\mu}]$. In particular $J_\mu(u_\mu)(u_\mu) = 0$, that is

$$(18) \quad \Phi'(u_\mu)(u_\mu) = \mu \int_0^T f(u_\mu(t)) u_\mu(t) dt,$$

for every $\mu \in (0, \bar{\mu}]$.

Then, by (18), it follows that

$$0 \leq 2|\cos(\pi\alpha)| \|u_\mu\|_\alpha^2 \leq \Phi'(u_\mu)(u_\mu) = \mu \int_\Omega f(u_\mu(t)) u_\mu(t) dt,$$

for any $\mu \in (0, \bar{\mu}]$. Letting $\mu \rightarrow 0^+$, by (17), we get $\lim_{\mu \rightarrow 0^+} \|u_\mu\|_\alpha = 0$, as claimed.

Finally, we show that the map

$$\mu \mapsto J_\mu(u_\mu) \text{ is strictly decreasing in } (0, \mu^*).$$

For this we observe that for any $u \in E_0^\alpha$, one has

$$(19) \quad J_\mu(u) = \mu \left(\frac{\Phi(u)}{\mu} - \Psi(u) \right).$$

Now, let us fix $0 < \mu_1 < \mu_2 \leq \bar{\mu} < \mu^*$ and let u_{μ_i} be the global minimum of the functional J_{μ_i} restricted to $\Phi((-\infty, r_{\bar{\mu}}))$ for $i = 1, 2$. Also, let

$$m_{\mu_i} := \left(\frac{\Phi(u_{\mu_i})}{\mu_i} - \Psi(u_{\mu_i}) \right) = \inf_{v \in \Phi^{-1}((-\infty, r_{\bar{\mu}}))} \left(\frac{\Phi(v)}{\mu_i} - \Psi(v) \right),$$

for every $i = 1, 2$.

Clearly, (15) together (19) and the positivity of μ imply that

$$(20) \quad m_{\mu_i} < 0, \quad \text{for } i = 1, 2.$$

Moreover,

$$(21) \quad m_{\mu_2} \leq m_{\mu_1},$$

thanks to the fact that $0 < \mu_1 < \mu_2$. Then, by (19)–(21) and again by the fact that $0 < \mu_1 < \mu_2$, we get that

$$J_{\mu_2}(u_{\mu_2}) = \mu_2 m_{\mu_2} \leq \mu_2 m_{\mu_1} < \mu_1 m_{\mu_1} = J_{\mu_1}(u_{\mu_1}),$$

so that the map $\mu \mapsto J_{\mu}(u_{\mu})$ is strictly decreasing in $(0, \bar{\mu})$.

The arbitrariness of $\bar{\mu} < \mu^*$ shows that $\mu \mapsto J_{\mu}(u_{\mu})$ is strictly decreasing in $(0, \mu^*)$.

We would like to note that Theorem 3.2 is a bifurcation result, since $\mu = 0$ is a bifurcation point for problem (F_f^{μ}) , in the sense that the pair $(0, 0)$ belongs to the closure of the set

$$\{(u_{\mu}, \mu) \in E_0^{\alpha} \times (0, +\infty) : u_{\mu} \text{ is a non-trivial weak solution of } (F_f^{\mu})\}$$

in $E_0^{\alpha} \times \mathbb{R}$.

Indeed, by Theorem 3.2 we have that

$$\|u_{\mu}\|_{\alpha} \rightarrow 0 \quad \text{as } \mu \rightarrow 0^+.$$

Hence, there exist two sequences $\{u_j\}$ in E_0^{α} and $\{\mu_j\}$ in \mathbb{R}^+ (here $u_j := u_{\mu_j}$) such that

$$\mu_j \rightarrow 0^+ \quad \text{and} \quad \|u_j\|_{\alpha} \rightarrow 0,$$

as $j \rightarrow +\infty$.

Moreover, we would like to stress that for any $\mu_1, \mu_2 \in (0, \mu^*)$, with $\mu_1 \neq \mu_2$, the solutions u_{μ_1} and u_{μ_2} given by Theorem 3.2 are different, thanks to the fact that the map

$$(0, \mu^*) \ni \mu \mapsto J_{\mu}(u_{\mu})$$

is strictly decreasing.

Remark 4.5. We just observe, for completeness, that Theorem 3.1 remains valid for equations like these

$$\frac{d}{dt} \left({}_0D_t^{\alpha-1} ({}_0^c D_t^{\alpha} u(t)) - {}_tD_T^{\alpha-1} ({}_t^c D_T^{\alpha} u(t)) \right) + f(t, u(t)) = 0, \quad \text{a.e. } t \in [0, T]$$

where $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function. In this case condition (S_G) assume the form

$$\sup_{\gamma > 0} \frac{\gamma^2}{\int_0^T \max_{|\xi| \leq \gamma} F(t, \xi) dt} > \frac{\kappa_\alpha}{T}, \quad (S_G^*)$$

where we set $F(t, \xi) := \int_0^\xi f(t, s) ds$, for every $(t, \xi) \in [0, T] \times \mathbb{R}$. Further, the conclusions of Theorem 3.2 are still true if we take

$$\mu \in \left(0, \frac{T}{\kappa_\alpha} \left(\sup_{\gamma > 0} \frac{\gamma^2}{\int_0^T \max_{|\xi| \leq \gamma} F(t, \xi) dt} \right) \right).$$

Finally, if $f(t, 0) \equiv 0$, we can assume, in order to obtain a non-trivial solution, that there are a non-empty open set $D \subseteq (0, T)$ and $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi \rightarrow 0^+} \frac{\operatorname{ess\,inf}_{t \in B} F(t, \xi)}{\xi^2} = +\infty,$$

and

$$\liminf_{\xi \rightarrow 0^+} \frac{\operatorname{ess\,inf}_{t \in D} F(t, \xi)}{\xi^2} > -\infty.$$

In conclusion, we consider a direct application of Theorem 3.2 and Remark 4.3.

Example 4.1. Let $\alpha \in (1/2, 1]$ and consider the following parametric problem (namely (F_g^μ)):

$$\begin{aligned} \frac{d}{dt} \left({}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u(t)) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u(t)) \right) + \mu g(u(t)) &= 0, \text{ a.e. } t \in [0, T] \\ u(0) = u(T) &= 0, \end{aligned}$$

where the non-linearity g has the form

$$(22) \quad g(u) := \begin{cases} u^{r-1} + u^{s-1} & \text{if } u \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and in which $1 < r < 2 < s$.

Owing to Theorem 3.2 and taking into account Remark 4.3 problem (F_g^μ) admits at least one non-trivial solution in E_0^α provided

$$0 < \mu < \frac{rs\bar{\gamma}^{2-r}}{\kappa_\alpha(s + r\bar{\gamma}^{s-r})},$$

where

$$\bar{\gamma} := \left(\frac{s(2-r)}{r(s-2)} \right)^{1/(s-r)}.$$

Moreover, from Remark 4.4, one also has

$$\lim_{\mu \rightarrow 0^+} \int_0^T |{}_0^c D_t^\alpha u_\mu(t)|^2 dt = 0,$$

and the function

$$\mu \mapsto - \int_0^T {}_0^c D_t^\alpha u_\mu(t) \cdot {}_t^c D_T^\alpha u_\mu(t) dt - \mu \int_0^T \left(\int_0^{u_\mu(t)} f(s) ds \right) dt,$$

is negative and strictly decreasing in $\left(0, \frac{rs\bar{\gamma}^{2-r}}{\kappa_\alpha(s + r\bar{\gamma}^{s-r})} \right)$.

Remark 4.6. We want to point out that the energy functional J_μ related to problem (F_g^μ) is not coercive. Indeed, fix $u \in E_0^\alpha \setminus \{0\}$ and let $\tau \in \mathbb{R}$. We have

$$\begin{aligned} J_\mu(\tau u) &= \Phi(\tau u) - \mu \int_0^T \left(\int_0^{\tau u(t)} g(s) ds \right) dt \\ &\leq \frac{\tau^2}{|\cos(\pi\alpha)|} \|u\|_\alpha^2 - \mu \frac{\tau^r}{r} \|u\|_{L^r}^r - \mu \frac{\tau^s}{s} \|u\|_{L^s}^s \rightarrow -\infty \end{aligned}$$

as $\tau \rightarrow +\infty$, bearing in mind that $r < 2 < s$.

Besides the papers [1, 2, 6, 4] and [7, 15, 34, 35, 37] we cite the monographs [11, 17, 18] as general references on the subject treated in this paper. See also [27, 29, 30, 31].

Acknowledgements. The authors are grateful to the referee for the careful analysis of this paper and for constructive remarks. The paper is realized with the auspices of the GNAMPA Project 2013 entitled: *Problemi non-locali di tipo Laplaciano frazionario*.

REFERENCES

- [1] R.P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. **109** (2010), 973-1033.

- [2] B. Ahmad and J.J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl. **58** (2009), 1838-1843.
- [3] C. Bai, *Positive solutions for nonlinear fractional differential equations with coefficient that changes sign*, Nonlinear Anal. **64** (2006), 677-685.
- [4] C. Bai, *Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative*, J. Math. Anal. Appl. **384** (2011), 211-231.
- [5] C. Bai, *Existence of solutions for a nonlinear fractional boundary value problem via a local minimum theorem*, Electron. J. Diff. Equ. **2012** (2012), 1-9.
- [6] Z. Bai and H. Lu, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311** (2005), 495-505.
- [7] M. Benchohra, S. Hamani and S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal. TMA **71** (2009), 2391-2396.
- [8] G. Bonanno and G. Molica Bisci, *Infinitely many solutions for a Dirichlet problem involving the p -Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A, **140**, no. 4, 1-16 (2010).
- [9] G. Borino, M. Di Paola and M. Zingales, *A non-local model of fractional heat conduction in rigid bodies*. Eur. J. Phys.-ST, in press, 2011
- [10] C.E. Eringen, *Theory of non-local thermoelasticity*. Int.J. Engng. Sci. **12** (1974), 1063-1077.
- [11] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
- [12] F. Jiao and Y. Zhou, *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Comput. Math. Appl. **62** (2011) 1181-1199.
- [13] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
- [14] A. Kristály, V. Rădulescu and Cs. Varga, *Variational principles in mathematical physics, geometry, and economics. Qualitative analysis of nonlinear equations and unilateral problems. With a foreword by Jean Mawhin*, *Encyclopedia of Mathematics and its Applications*, 136, Cambridge University Press, Cambridge (2010).
- [15] V. Lakshmikantham and A.S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. TMA **69** (2008), 2677-2682.
- [16] J. Mawhin and M. Willem, *Critical Point Theorey and Hamiltonian Systems*, Springer, New York, 1989.
- [17] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
- [18] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [19] Y. Z. Povstenko, *Theory of termoelasticiy based on the space-time-fractional heat conduction equation*. Phys. Scr. (2009) T136, 014017.
- [20] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, in: CBMS, vol. **65**, American Mathematical Society, 1986.

- [21] F. Riewe, *Nonconservative Lagrangian and Hamiltonian mechanics*, Phys. Rev. E **53** (1996), 1890-1899.
 - [22] B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math. **113** (2000), 401-410.
 - [23] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integral and Derivatives: Theory and Applications*, Gordon and Breach, Longhorne, PA, 1993.
 - [24] R. Servadei, *The Yamabe equation in a non-local setting*, to appear in Adv. Nonlinear Anal. **3**, (2013).
 - [25] R. Servadei, *A critical fractional Laplace equation in the resonant case*, to appear in Topol. Methods Nonlinear Anal.
 - [26] R. Servadei, *Infinitely many solutions for fractional Laplace equations with subcritical nonlinearity*, to appear in Contemp. Math.
 - [27] R. Servadei and E. Valdinoci, *Lewy-Stampacchia type estimates for variational inequalities driven by nonlocal operators*, to appear in Rev. Mat. Iberoam. **29** (2013).
 - [28] R. Servadei and E. Valdinoci, *Weak and viscosity solutions of the fractional Laplace equation*, to appear in Publ. Mat.
 - [29] R. Servadei and E. Valdinoci, *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. **389** (2012), 887-898.
 - [30] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33**, 5 (2013), 2105-2137.
 - [31] R. Servadei and E. Valdinoci, *The Brézis-Nirenberg result for the fractional Laplacian*, to appear in Trans. AMS.
 - [32] H. H. Sherief, A.M.A. El-Sayed, A.M. Abd El-Latief, *Fractional order theory of thermoelasticity*. Int. J. Sol. Str. (2010), **47**, 269-275.
 - [33] M. Struwe, *Variational methods, Applications to nonlinear partial differential equations and Hamiltonian systems*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, Springer Verlag, Berlin-Heidelberg (1990).
 - [34] J. Wang and Y. Zhou, *A class of fractional evolution equations and optimal controls*, Nonlinear Anal. RWA **12** (2011), 262-272.
 - [35] Z. Wei, W. Dong and J. Che, *Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative*, Nonlinear Anal. **73** (2010), 3232-3238.
 - [36] M. Willem, *Minimax theorems*, Birkhäuser, 1996.
 - [37] S. Zhang, *Positive solutions to singular boundary value problem for nonlinear fractional differential equation*, Comput. Math. Appl. **59** (2010), 1300-1309.
- E-mail address:* marek.galewski@p.lodz.pl
E-mail address: gmolica@unirc.it

(M. Galewski) INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF LODZ,
 WOLCZANSKA 215, 90-924 LODZ, POLAND

(G. Molica Bisci) DIPARTIMENTO P.A.U., UNIVERSITÀ DEGLI STUDI "MEDITERRANEA" DI REGGIO CALABRIA, SALITA MELISSARI - FEO DI VITO, 89100 REGGIO CALABRIA, ITALY